Lecture 5: Rigid Bodies

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3.1 Introduction

concurrent forces

A **body** is a combination of a **large number of particles**.

- so the size of the body should be taken into consideration
- the **point of application** of different forces will act on different particles

A **rigid body** is defined as one that <u>does not deform</u>

- actual machines *do* deform under loads, but the deformations are insignificant

Two important concepts are associated with the effect of a force on a rigid body:

- ❖ the moment of a force about a point
- ❖ the moment of a force about an axis

We'll also discuss a **couple**, which is a combination of 2 forces that have the **same magnitude**, **parallel lines of action**, and **opposite sense**.

3.2 External and internal forces

Forces acting on rigid bodies can be separated into 2 groups:

- **External forces** represent the action of other bodies on the rigid body
  - entirely responsible for external behaviour of rigid body
  - **will either cause it to move or ensure it remains at rest**
- **Internal forces** hold together the particles of the rigid body
  - this also includes forces holding separate parts of a rigid body together

3.3 Principle of Transmissibility

Principle of transmissibility states that conditions of equilibrium or motion of a rigid body will remain unchanged if a force **F** acting at a given point of the rigid body is replaced by a force **F** of the *same magnitude*, *same direction*, but acting at a *different point*, provided the 2 forces have the *same line of action*.

- based on experimental evidence and can't be derived from previously discussed properties

Statics of rigid bodies will be based on these 3 principles:

- parallelogram law of addition
- Newton's first law
- principle of transmissibility

→ the point of application of the force does not matter, <u>as long as the lines of action remain unchanged</u>

→ forces acting on a rigid body represented by a different kind of vector, called a **sliding vector**

3.4 Vector product of two vectors

To understand the effect of a force on a rigid body, the **moment of a force about a point** is extremely significant. First, we need to understand the vector product of two vectors:

The vector product of 2 vectors, **P** and **Q**, is defined by vector **V**

- line of action of **V** is perpendicular to the plane containing **P** and **Q**
- magnitude of **V** is product of the magnitudes of **P** and **Q** and multiplied by the sine of the angle theta formed by **P** and **Q** (always 180 degrees or less)
  $V = PQsin(\theta)$
  - θ is the angle between **P** and **Q**
  - from this equation, if **P** and **Q** have same or opposite directions the

vector product is 0

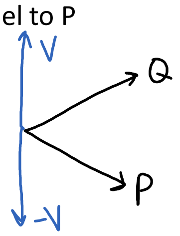
- direction of **V** is obtained from right hand rule
  - *The right-hand rule states that the orientation of the vectors' cross product is determined by placing u and v tail-to-tail, flattening the right hand, extending it in the direction of u, and then curling the fingers in the direction that the angle v makes with u. The thumb then points in the direction of u×v.* (http://mathworld.wolfram.com/Right-HandRule.html)
- if **P** and **Q** don't have the same point of application, they need to be redrawn first at the same point
- **P**, **Q**, and **V** (in that order) form a **right-handed triad**

V is the **vector** product (cross product) of **P** and **Q**:

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q}$$

◇ **P** x **Q** will remain unchanged if **Q** is replaced with **Q'**, which is coplanar with **Q**, and the tips of **Q** and **Q'** form a line parallel to P

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q} = \mathbf{P} \times \mathbf{Q}'$$



◇ The vector product is not commutative

$$(\mathbf{Q} \times \mathbf{P}) = -(\mathbf{P} \times \mathbf{Q})$$

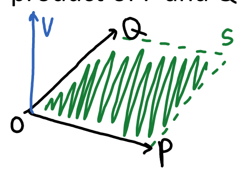
or associative:

$$(\mathbf{P} \times \mathbf{Q}) * \mathbf{S} \neq \mathbf{P} \times (\mathbf{Q} \times \mathbf{S})$$

But it *is* distributive

$$\mathbf{P} \times (\mathbf{Q_1} + \mathbf{Q_2}) = \mathbf{P} \times \mathbf{Q_1} + \mathbf{P} \times \mathbf{Q_2}$$

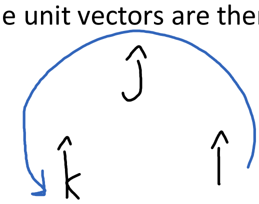
The magnitude V of the vector product of P and Q is equal to the area of OQSP.



3.5 Vector products expressed in terms of rectangular components

The unit vectors i, j, and k are mutually perpendicular and form a right-handed triad. The cross products of the unit vectors are therefore:

i x **i** = 0	**j** x **i** = -k	**k** x **i** = j
i x **j** = k	**j** x **j** = 0	**k** x **j** = -i
i x **k** = -j	**j** x **k** = i	**k** x **k** = 0



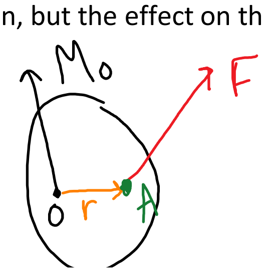
To easily determine the sign of the cross product of the unit vectors, use this diagram. The product is positive in the counter clockwise direction, and negative in the clockwise.

The rectangular components of the vector product **V** are:

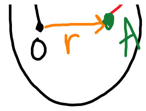
$$V_x = P_y Q_z - P_z Q_y$$
$$V_y = P_z Q_x - P_x Q_z$$
$$V_x = P_x Q_y - P_y Q_x$$

3.6 Moment of a force about a point

Force **F** acts on a rigid body. **F** is represented by a vector defining its magnitude and direction, but the effect on the rigid body depends on its point of application A.



The position of A is defined by the vector r which joins a fixed reference point, O, with A. (r is the position vector of A)



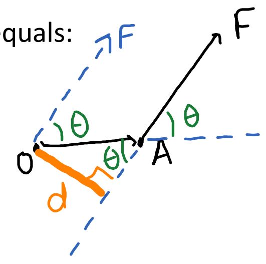
The moment of **F** about O is defined as the vector product of r and **F**:

$\mathbf{M_O} = r\,x\,\boldsymbol{F}$

- $\mathbf{M_O}$ is perpendicular to both r and **F**
- the sense of $\mathbf{M_O}$ is defined by the sense of rotation which will bring the vector r in line with the vector **F** - this rotation is observed as CCW from the tip of $\mathbf{M_O}$
  - this can also be found using the right hand rule, described earlier (the thumb indicates the sense of the moment M_O

A moment about a point also equals:
$\mathbf{M_O} = rF\sin\theta = Fd$
where d is the perpendicular distance from O to the line of action of **F**.

$d = \frac{M_o}{F}$



The magnitude of $\mathbf{M_O}$ measures the tendency of the force **F** to make the rigid body rotate about a fixed axis directed along $\mathbf{M_O}$, and is measured in **newton-metres (N·m)**.
Although the moment of a force depends on its magnitude, line of action, and its sense, it **does not** depend on the actual position of the point of application of the force along its line of action. The line of action of **F** must lie in a plane through O and perpendicular to the moment $\mathbf{M_O}$.

The principle of transmissibility states that **F** and **F'** have the same effect on a rigid body if they have the same magnitude, direction, and line of action.

- alternatively, **F** and **F'** are only equivalent if and only if they are equal (same magnitude and direction) and have **equal moments about a given point O**.

Therefore, in order for two forces **F** and **F'** to be equivalent these conditions must be satisfied:

$\mathbf{F} = \mathbf{F}'$	$\mathbf{M_O} = \mathbf{M_O}'$

NOTE: when describing the moment of a force, counter-clockwise is generally taken as positive and clockwise negative.

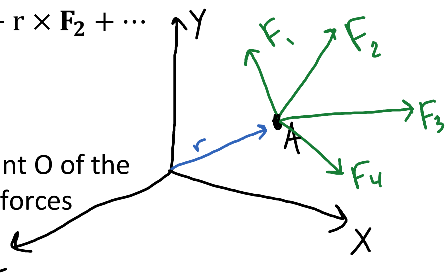
3.7 Varignon's Theorem

The **distributive property** of vector products can be used to determine the moment of the resultant of several concurrent forces.
ie. if forces $\mathbf{F_1}$, $\mathbf{F_2}$, ... are applied at point A, and r is the position vector of A:

$\mathrm{r} \times (\mathbf{F_1} + \mathbf{F_2} + \cdots) = \mathrm{r} \times \mathbf{F_1} + \mathrm{r} \times \mathbf{F_2} + \cdots$

or

$\mathbf{M_O} = \Sigma \mathbf{M_{O,i}}$



"The moment about a given point O of the resultant of several concurrent forces is equal to the sum of the moments of the various forces about the same point O"

3.8 Rectangular components of the moment of a force

Resolving r and **F** into rectangular x, y, and z components simplifies the determination of a force in space.

$r = \mathrm{x}\mathbf{i} + \mathrm{y}\mathbf{j} + \mathrm{z}\mathbf{k}$

$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$

$\mathbf{M_O} = M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k} \quad = \quad \mathrm{r} \times \mathbf{F}$